Abstract

The model presented in this paper aims to provide a powerful insight on the classic Spatial Iterated Prisoners Dilemma game, and how an invading wave of Tit-for-Tat (TFT) individuals could invade a hostile population dominated by an Always-Defect (AD) strategy. This particular form of the game has previously been studied in a one-dimensional system. The main contribution of this paper is an adaptation to provide a 2-dimensional version that can be reproduced and analyzed. As well as a comparison between two different initial density distribution of the two strategies. Special attention is allocated to the individuals' mobility, determined by a simple diffusion process, and its influence upon TFT’s success when invading. The model proved TFT’s need for a higher mobility in order to defeat AD’s hostility as the wave diffuses out. Some intriguing results were also obtained when starting from a big cluster of TFT, and a co-existence of both strategies was observed.

1 Introduction

The emergence of bigger and more demanding global problems has lead to a greater interest in the study of behavioral economics, which is the study of multiple games [1] [2] that portray the process of decision making as an attempt to explain and understand the nature of not only human but biological interactions between and within species [3] [4] [5]. Starvation in poor countries, violation of Human Rights and climate change are just a few examples that illustrate the magnitude of these global problems beset humanity.

This paper provides an insight of a particular game called "The Prisoners Dilemma” in which two players are presented with two options: cooperate or defect. A matrix of payoffs determines the different outcomes of each player. Tit for Tat (TFT) and Always Defect (AD) are the two strategies analyzed throughout this paper. These were first tested during a tournament [6] and the first one won the two times the simulation was ran.

The Iterated Spatial Prisoners Dilemma (ISPM) is that which takes place many times, with repeated interactions, and is represented in a spatial system. Previous analysis on TFT and AD have been made in a 1-Dimensional system through a simple diffusion model, which have shown that small clusters of TFT can succeed in a hostile environment dominated by ADs [7].

The main contribution of this paper is an adaptation of the model presented by Ferriere and Michod [7] into a 2-Dimensional system, as well as a comparison of TFT’s performance in two different spatial initial conditions: TFT in a big cluster and TFT distributed randomly.

The rest of the paper is organized as follows. In Section 2 I introduce the model used in this study, its characteristics and parameters. In Sec-
tion 3 I present the results divided into 2 sub-
sections according to the spatial initial condi-
tions, and 2 subsubsections according to AD’s
and TFT’s mobilities. In Section 4 I present the
discussion and conclusions reached. Section 5
provides some information about possible appli-
cations.

2 Methods

The Prisoners Dilemma is a game in which two
players face the option to cooperate (C) or defect
(D) and the pay-offs assigned to each player are
illustrated in Table 1:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>R,R</td>
<td>S/T</td>
</tr>
<tr>
<td>D</td>
<td>T,S</td>
<td>P,P</td>
</tr>
</tbody>
</table>

Table 1: Matrix of payoffs for the general pris-
oners Dilemma

These pay-offs are in terms of costs and be-
nefits: $T = 1 + b, R = 1 + b - c, P = 1$ and $S = 1 - c$,
where $c > 0$ and $b > c$. This yields the follow-
ing relation: $T > R > P > S$. This in turn can
be expressed as: $2R > S + T$ which means that
the communal reward of mutual cooperating is
greater than the communal reward of mutual de-
fection.

The strategies considered have specific behav-
ior that is imbedded in their identity (or code).
AD is programmed to defect at all times. TFT
cooperates at first and then it acts according to
the opponent’s past behavior. If the opponent
has defected them previously, TFT will retali-
ate and defect back. On the other hand if the
opponent is another TFT, based on previous be-
havior they will always cooperate between each
other. To simplify the model it is assumed that
TFT has a limited memory, where $w$ is the pro-
bability that an individual TFT will recognize its
opponent from previous interactions. Other pa-
rameters that are also included in the analysis
are the interaction time ($\tau$) and the death rate
($d$).

All these parameters must be reflected in the
payoffs of both TFT and AD. For $\tau$ and $d$, the
payoffs are multiplied by a factor of $e^{-dr}$. For
$w$, the payoffs vary depending on the strategies.
For example, if TFT was to face AD, the payoffs
would be as follows: $TFT = wP + (1 - w)S$ and
$AD = wP + (1 - w)T$.

Mobility of individuals is expressed in terms of
$\mu, \mu_{TFT}$ for TFT and $\mu_{AD}$ for AD. Movement is
given by a simple diffusion equation, where indi-
viduals randomly walk around a 2-Dimensional
grid. The probability that an individual will
cover a distance $\xi$ in $\Delta$ units of time is given
by the Gaussian distribution: $\frac{e^{-\xi^2/4\mu\Delta t}}{2\sqrt{\pi\mu\Delta t}}$.

Mobility takes place in a squared grid of length
$L$, which is un turn subdivided into discrete con-
tiguous cells of length $a = 2/K$, where $K$ is the
carrying capacity. Based on the mobilities and
resulting payoffs from the various interactions
TFT’s and AD’s density change over time and
through space. Density $n$, $n_{TFT}$ for TFT and
$n_{AD}$, is a function of both time and space ($x,t$).
This is expressed in the following equations:

$$\frac{\partial n_{TFT}}{\partial t} = n_{TFT}\left[\frac{(An)_{TFT}}{n}\right] - \left[n^*\frac{An}{n^2}\right]
+ \mu_{TFT}\left[\frac{\partial^2 n_{TFT}}{\partial x^2}\right]$$

$$\frac{\partial n_{AD}}{\partial t} = n_{AD}\left[\frac{(An)_{AD}}{n}\right] - \left[n^*\frac{An}{n^2}\right]
+ \mu_{AD}\left[\frac{\partial^2 n_{AD}}{\partial x^2}\right]$$

Where $A$ represents the matrix of payoffs that
constitutes the ISPD:

$$A = \begin{pmatrix} (e^{-dr})R & (e^{-dr})[wP + (1 - w)S] \\ (e^{-dr})[wP + (1 - w)T] & (e^{-dr})P \end{pmatrix}$$

Eq. (1) and Eq. (2) are the matrix form of the
equations used. In order to further understand
the mobility and change in density for both TFT and AD, Eq. (3) must be plugged into them. The following equations show the final result of this:

\[
\frac{dn_{TFT}}{dt} = \frac{1}{n} \left[ (e^{-\alpha t} / \tau)n_{TFT}^2 + (e^{-\alpha t} / \tau)n_{AD}\tau + (1 - w)S_{n_{TFT}n_{AD}} \right] \\
- \frac{1}{n} \left[ (e^{-\alpha t} / \tau)n_{AD}^2 + (e^{-\alpha t} / \tau)n_{TFT}\tau + (1 - w)S_{n_{TFT}n_{AD}} \right] \\
+ (e^{-\alpha t} / \tau)(w + (1 - w)T)n_{AD}^2 + (e^{-\alpha t} / \tau)n_{TFT}^2 + \mu_{TFT} \frac{dn_{TFT}}{dx^2} 
\]

\[
\frac{dn_{AD}}{dt} = \frac{1}{n} \left[ (e^{-\alpha t} / \tau)n_{AD}^2 + (e^{-\alpha t} / \tau)n_{TFT}\tau + (1 - w)S_{n_{TFT}n_{AD}} \right] \\
- \frac{1}{n} \left[ (e^{-\alpha t} / \tau)n_{AD}^2 + (e^{-\alpha t} / \tau)n_{TFT}\tau + (1 - w)S_{n_{TFT}n_{AD}} \right] \\
+ (e^{-\alpha t} / \tau)(w + (1 - w)T)n_{AD}^2 + (e^{-\alpha t} / \tau)n_{TFT}^2 + \mu_{AD} \frac{dn_{AD}}{dx^2} 
\]

Eq. (4) and Eq. (5) were the principal component of the code used to simulate the diffusion system. For each equation, the first two terms represented a Lotka-Volterra interaction between TFT and AD, also known as predator-prey equations (LOOK FOR REFERENCES THAT EXPLAIN LV). The third and last term of the equations represents the diffusion component of the code.

Some assumptions had to be considered in this model, these were taken from the original model in 1-Dimension [7]:

1. \( k \) is constant and it only fluctuates around \( \tau \). This because during the time of interaction there are some population adjustments that have an effect on \( k \) but which are negligible in terms of time.

2. \( k \) is large so that the model is continuous in space.

3. \( \tau \) must be small for the model to be continuous in time

4. The game’s payoffs are in terms of the individual’s reproductive success, not survivorship

3 Results

The analysis was first divided in terms of the initial conditions for distribution, one that had random distribution all over the grid for both TFT and AD, and another that had a cluster in the middle with a high initial density of TFT and everywhere else a random distribution of both strategies. Within each of these, two other parameters where modified to yield different analysis, one in which AD’s mobility is greater than TFT’s and another one with TFT’s mobility greater than AD’s.

All the different scenarios were just in terms of the initial conditions. After these were set, in all cases the same code was run with all other parameters kept the same. This meant that the results following this would be in terms of the interaction between TFT and AD, as well as the distribution of both strategies across space as time passed by.

The parameters taken for all runs, besides the ones modified as the study’s subject, were assigned according to Table 2.

| \( \lambda \) | 0.25 |
| \( \mu \) | 0.5 |
| \( \sigma \) | 0 |
| \( \tau \) | 2 |
| \( \beta \) | 5 |

Table 2: Constant parameters for all runs

3.1 Random distribution

For this part of the analysis, the distribution of TFT and AD was randomly generated by the computer all over the grid.

3.1.1 AD’s mobility greater than TFT’s [\( \mu_{AD} > \mu_{TFT} \)]

The first variation when having initial distribution randomly assigned all over the grid was to make AD’s mobility greater than TFT’s, shown in Table(3) The results observed are shown in Fig. (1).

What Fig. (1) shows the change in \( n_{AD} \) as time goes by, from the left to right. These images show how under Random distribution and with AD having the highest mobility, AD will succeed and overpopulate the space. This can
Table 3: Different mobilities for first run of the Random distribution

<table>
<thead>
<tr>
<th>Mobilities</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>AD</td>
<td>0.4</td>
</tr>
<tr>
<td>TFT</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Figure 1: \( n_{AD} \) in random distribution with \( \mu_{AD} > \mu_{TFT} \). These pictures represent how \( n_{AD} \) (AD’s density) changes as time passes. Where blue is less dense and red is the highly concentrated AD population

be explained by the fact that with greater mobility AD (0.4) can easily take advantage of the limited TFT’s memory, which means that AD would be getting the higher payoff and having a greater reproduction success.

3.1.2 TFT’s mobility greater than AD’s \( [\mu_{TFT} > \mu_{AD}] \)

Afterwards, the mobilities where modified in such way that TFT’s was greater than AD’s, according to Table(4). Result of how TFT’s density behaved is shown in Fig.(2).

Table 4: Different mobilities for second run of the random distribution

<table>
<thead>
<tr>
<th>Mobilities</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>AD</td>
<td>1.3</td>
</tr>
<tr>
<td>TFT</td>
<td>1.7</td>
</tr>
</tbody>
</table>

Figure 2: \( n_{TFT} \) in random distribution with \( \mu_{TFT} > \mu_{AD} \). These pictures represent how \( n_{TFT} \) (TFT’s density) changes as time passes. Where blue is less dense and red is the highly concentrated TFT population

What Fig. (2) shows is the change in \( n_{TFT} \) as time goes by, from the left to the right. These images show how under random distribution and with TFT having the highest mobility (1.7), TFT will succeed and overpopulate the space. These results were already observed and discussed [7], and it is basically justified by the following circumstances.

Pioneering TFT, who are at the front of the wave invasion, should be able to survive AD’s hostility. There are two ways for these to happen:

1. If the venturing TFTs move together with other TFTs and they mutually benefit from their cooperative nature

2. If the venturing TFTs move together with a known AD that can be remembered from previous encounters and therefore mutual Defection would not result in a negative outcome for the pioneering TFT

Intrusive AD penetrating small TFT populations should not damage TFT’s cooperative environment and take advantage of it, by making TFT worse off and decreasing their reproduction success. These can be avoided under the following circumstances:

1. Intrusive AD moving together with another AD, so that they mutually defect and do not affect TFT’s reproductive success

2. Intrusive AD move into the small TFT populations together with a previously encountered TFT individual. This would guarantee that the TFT would remember the AD and defect back, which would also mean a
better result that an unknown AD defecting to an innocent TFT

3.2 Clusters included in the initial density distribution \( [n_{AD} and n_{TFT}] \)

After the previous results which were consistent with results discussed before, a second scenario was built were the initial density distribution included a big cluster of either TFTs or ADs at the center of the grid. The purpose of this was to investigate whether having clusters of either strategy would influence the way densities developed as times passed and interactions took place.

3.2.1 AD’s mobility greater than TFT’s \( [\mu_{AD} > \mu_{TFT}] \)

The first variation within the clustered initial distribution was to make AD’s mobility greater than TFT’s in order to compare this to the non-clustered version. The mobilities used are the same shown in Table(3) and the results observed are shown in Fig. (3) and Fig. (4).

![Figure 3: n_{AD}’s progression when starting from a random initial density distribution except from a very dense cluster of TFTs at the center of the grid, and with \( \mu_{AD} > \mu_{TFT} \). Blue is less dense and red is the highly concentrated AD population.](image)

In this case, the figures for \( n_{AD} \) and \( n_{TFT} \) are included because the results show some interesting features in both cases. For the previous cases, with a fully random distribution, either TFT or AD density increased by making the other one decrease. However, when starting with a cluster of TFT, the result was that both densities increased (going from blue to red everywhere). This appeared to be a disconcerting event since it was expected that whichever strategy with the highest mobility would take over and increase its density by making the other one decrease.

After changing many of the parameters, and running the program many times it was decided that it wasn’t a coincidental result driven by a coding error, but an intriguing result that should be taken into consideration. Fortunately enough, and after some time researching, previous literature had already discussed this behavior [8].

3.2.2 TFT’s mobility greater than AD’s \( [\mu_{TFT} > \mu_{AD}] \)

The second variation within the clustered initial distribution was to make TFT’s mobility greater than AD’s in order to compare this to the non-clustered version. The mobilities used are the same shown in Table(4) and the results observed are shown in Fig. (5) and Fig. (6).

![Figure 4: n_{TFT}’s progression when starting from a random initial distribution except from a very dense cluster of itself at the center of the grid, and with \( \mu_{AD} > \mu_{TFT} \). Blue is less dense and red is the highly concentrated TFT population.](image)

Just like in the previous example, we observe a different behavior than the expected. In this cases, just as Fig.(6) shows, there was an increase in TFT’s density \( (n_{TFT}) \) up to the highest levels. There is a consistent increase in its density that spreads out from the very little populations scattered around the grid as well as from the dense
Figure 5: $n_{AD}$’s progression when starting from a random initial density distribution except from a very dense cluster of TFTs at the center of the grid, and with $\mu_{TFT} > \mu_{AD}$. Blue is less dense and red is the highly concentrated AD population.

On the other hand, Fig. (5) shows a very interesting behavior in terms of AD’s density ($n_{AD}$) evolution. It starts with a random initial density distribution and then it starts growing in patches around the TFT cluster in the center. AD is never strong enough to penetrate the cluster, but instead it started forming medium sized clusters around which stabilized after some time. The stable state that is reached by AD is shown in the last square of Fig.(5).

This is, just like in the previous example, a non-expected behavior. This coexistence between TFT and AD, had previously been observed [8], but it a stable equilibrium like the one found here is not specifically expressed in any found literature. It may be that when TFT starts from a cluster and has an increased mobility; it is strong enough to keep AD controlled but not to completely eliminate it.

4 Discussion and Conclusion

In the end, four different variations where made, and the results are synthesized in Table(5).

<table>
<thead>
<tr>
<th>Initial distribution</th>
<th>Mobilities</th>
<th>Dominant strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete random distribution</td>
<td>AD &gt; TFT</td>
<td>AD</td>
</tr>
<tr>
<td>TFT &gt; AD</td>
<td>TFT</td>
<td></td>
</tr>
<tr>
<td>TFT cluster + random distribution</td>
<td>AD &gt; TFT</td>
<td>AD and TFT</td>
</tr>
<tr>
<td>TFT &gt; AD</td>
<td>AD and TFT</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: This table shows the different outcomes for the the various runs that were made, each with the different parameters and circumstance previously mentioned.

The first observation that can be made is that there is a notable difference in the outcome between the two initial density distributions. When there was a random distribution all over the grid there was a clear dominant strategy, either TFT or AD, depending on the mobilities. Whereas in the clustered version unexpected behavior was observed, and coexistence between TFT and AD arose. One explanation for this is that cluster distributions interfere and modify the interaction term of the equation (Lotka Volterra), which is expressed as the first and second term of Eq. (2) and Eq. (1).

The following points summarize the results encountered and the conclusions that were drawn:

1. When functioning in a non-clustered system, mobilities have a great influence upon the succeeding strategy.
2. When functioning in a system with clusters, mobilities have little or no influence upon the succeeding strategy given that the
strategies will reach an equilibrium were they can coexist.

The results observed where carefully ran and recorded however there are some limitations of the model. The major one is that the model used was taken from a 1-D diffusion system. For further research it would be great to try and make a more accurate translation of this.

The current diffusion method is a good approximation for the Iterated Spatial Prisoners Dilemma, however to have a better outcome it would be ideal to use a Principal-Agent model that could track all of the individuals closely. This type of analysis would yield more accurate results.

Some further research would not only include improvements to mathematical model, but also about its application. An example of how this could be taken further is by applying the ISPM to climate change negotiations. Some adaptations have already been done [9] [10] [11] but coding this and modeling it would be something interesting to do. Just a quick overview of how the matrix would look like is shown in Table(6) [12].

<table>
<thead>
<tr>
<th>Country A</th>
<th>Country B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mitigate and adapt</td>
<td>Mitigate and adapt</td>
</tr>
<tr>
<td>C·T(0), C·T(0)</td>
<td>C·T(1),T(1)</td>
</tr>
<tr>
<td>No action</td>
<td>T(1),T(1),T(4)</td>
</tr>
</tbody>
</table>

Table 6: This matrix shows the payoffs that could be considered when applying the ISPM to Climate Change negotiations. Where c is the cost of mitigation and T(0), T(1) and T(4) are the costs of a global rise temperature of 1, 2 and 4 degrees respectively.

Just like this example, many more applications can be derived, not only in Economics [13], but also in Biology [3] and [2], Politics [14], and many more disciplines. I believe that the study of human relationships and interactions is extremely important given the complexity of the current global problems that are gaining more and more significance.

Acknowledgments
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References

